

CLASSIFICATION OF GLOBALLY F-REGULAR F -SANDWICHES OF HIRZEBRUCH SURFACES

TADAKAZU SAWADA

ABSTRACT. Let X be a smooth variety over an algebraically closed field of positive characteristic. An F -sandwich of X is a normal variety Y through which the relative Frobenius morphism of X factors as $F : X \rightarrow Y \rightarrow X$. In this paper, we give a classification of globally F-regular F -sandwiches of Hirzebruch surfaces.

INTRODUCTION

Let X be a smooth variety over an algebraically closed field of positive characteristic. A Frobenius sandwich of X is a normal variety Y through which the (iterated) relative Frobenius morphism of X factors as $F : X \rightarrow Y \rightarrow X$. For a given variety X , it is natural to ask what kinds of singularities and varieties appear as Frobenius sandwiches of X . However, it seems hopeless to classify the Frobenius sandwiches explicitly without any restriction to those under consideration because of pathological phenomena in positive characteristic. For example, there exists a Frobenius sandwich of the projective plane \mathbb{P}^2 whose nonsingular model is a surface of general type. (In characteristic 0, every unirational surface is rational.) Taking into account such pathological phenomena, we consider Frobenius sandwiches that behave better in the sense of Frobenius splitting, that is, globally F-regular Frobenius sandwiches. Global F-regularity is defined via splitting of Frobenius morphisms and has remarkably nice properties. Assuming global F-regularity, we expect to exclude pathological cases, so that we may hope for comprehensive study of Frobenius sandwiches.

We consider the following problem:

Problem. Given a globally F-regular variety X , classify globally F-regular Frobenius sandwiches of X .

We dealt with the simplest case where $X = \mathbb{P}^2$ in [7]. We showed that globally F-regular F -sandwiches of \mathbb{P}^2 of degree p (see Section 1 for the definition of degree) are singular toric surfaces and there are $p - 1$ isomorphism classes. In this paper, we give a classification of globally F-regular F -sandwiches of Hirzebruch surfaces of degree p . In

particular, we see that those globally F -regular F -sandwiches are toric surfaces and there are p or $p+1$ isomorphism classes for each Hirzebruch surface. The following is the main result:

Theorem. *A globally F -regular F -sandwich of the Hirzebruch surface \mathcal{H}_d of degree p is isomorphic to either one of the toric surfaces $X_{\Sigma_{di}}$ ($0 \leq i \leq p$).*

Here $X_{\Sigma_{di}}$ stands for the toric surface associated to a fan Σ_{di} . The fans Σ_{di} are given by considering the original fan on $N = \mathbb{Z}^2$ associated to \mathcal{H}_d on an overlattice $N + \mathbb{Z}_p^1(1, i)$ ($0 \leq i \leq p$). Hence we easily see that $X_{\Sigma_{di}}$ are globally F -regular F -sandwiches of \mathcal{H}_d of degree p . Conversely, this theorem says that globally F -regular F -sandwiches of \mathcal{H}_d of degree p are only those natural ones.

In Section 1, we review generalities on Frobenius sandwiches and globally F -regular varieties. In Section 2, we give the classification.

1. PRELIMINARY

We work over an algebraically closed field k of characteristic $p > 0$. Let X be an algebraic variety over k . The absolute Frobenius morphism $F : X \rightarrow X$ is the identity on the underlying topological space of X , and the p -th power map on the structure sheaf \mathcal{O}_X , which we also denote by $F : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$. Let $X^{(-1)}$ be the base change of X by the absolute Frobenius morphism of $\mathrm{Spec} k$. The relative Frobenius morphism $F_{\mathrm{rel}} : X \rightarrow X^{(-1)}$ is defined by the following Cartesian square:

$$\begin{array}{ccccc} & & F & & \\ & \nearrow & & \searrow & \\ X & \xrightarrow{F_{\mathrm{rel}}} & X^{(-1)} & \xrightarrow{\quad} & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec} k & \xrightarrow{F} & \mathrm{Spec} k \end{array}$$

In what follows, we use these variants of Frobenius morphisms interchangeably. Since we work over the algebraically closed field, we need not strictly distinguish these variants.

Frobenius sandwiches. First we review generalities on Frobenius sandwiches. Let X be a smooth variety over k . A normal variety Y is an F^e -sandwich of X if the e -th iterated relative Frobenius morphism

of X factors as

$$\begin{array}{ccc} X & \xrightarrow{F_{\text{rel}}^e} & X^{(-e)} \\ & \searrow \pi & \nearrow \rho \\ & Y & \end{array}$$

for some finite k -morphisms $\pi : X \rightarrow Y$ and $\rho : Y \rightarrow X^{(-e)}$, which are homeomorphisms in the Zariski topology. An F -sandwich will mean an F^1 -sandwich. We say that the Frobenius sandwich Y is of *degree* p if the degree of the morphism $\pi : X \rightarrow Y$ is p .

By a 1-*foliation* of X , we mean a saturated p -closed subsheaf L of the tangent bundle T_X closed under Lie brackets, where L is said to be p -closed if it is closed under p -times iterated composite of differential operators; see [4].

It is known that there are one-to-one correspondences among the followings (see [10], [4], [8]):

- F -sandwiches of X of degree p ;
- invertible 1-foliations of X ;
- p -closed rational vector fields of X modulo an equivalence \sim .

(1) F -sandwiches and 1-foliations: The correspondence is given by

$$Y \mapsto L = \{\delta \in T_X \mid \delta(f) = 0 \text{ for all } f \in \mathcal{O}_Y\} \subset T_X$$

and

$$L \mapsto \mathcal{O}_Y = \{f \in \mathcal{O}_X \mid \delta(f) = 0 \text{ for all } \delta \in L\} \subset \mathcal{O}_X.$$

(The inclusion $\mathcal{O}_Y \subset \mathcal{O}_X$ induces the finite morphism $\pi : X \rightarrow Y$.) The well-definedness of this correspondence is guaranteed by the Galois correspondence due to Aramova and Avramov [1].

(2) F -sandwiches and rational vector fields: We define an equivalence relation \sim between rational vector fields $\delta, \delta' \in \text{Der}_k k(X)$ as follows: $\delta \sim \delta'$ if and only if there exists a non-zero rational function $\alpha \in k(X)$ such that $\delta = \alpha\delta'$. Let $\{U_i = \text{Spec } R_i\}_i$ be an affine open covering of X . Given a p -closed rational vector field $\delta \in \text{Der}_k k(X)$, we have a quotient variety X/δ defined by glueing $\text{Spec } R_i^\delta$, where $R_i^\delta = \{r \in R_i \mid \delta(r) = 0\}$, and a quotient map $\pi_\delta : X \rightarrow X/\delta$ induced from the inclusions $R_i^\delta \subset R_i$. Then we easily see that R_i^δ is normal and the field extension $\text{Frac } R_i / \text{Frac } R_i^\delta$ is purely inseparable of degree p . This means that X/δ is an F -sandwich of degree p with the finite morphism $\pi_\delta : X \rightarrow X/\delta$ through which the Frobenius morphism of X factors. Conversely, if Y is an F -sandwich of X of degree p with the finite morphism $\pi : X \rightarrow Y$ through which the Frobenius morphism of X factors, then there exists a rational vector field δ such that $\pi = \pi_\delta$ and $Y = X/\delta$. Indeed, there exists a p -closed rational vector field

$\delta \in \text{Der}_k k(X)$ such that $k(X)^\delta = k(Y)$ by Baer's result (see e.g., [9]), since the field extension $k(X)/k(Y)$ is purely inseparable of degree p . Thus δ induces an inclusion $\mathcal{O}_Y \subset \mathcal{O}_{X/\delta}$, so that there exists a finite birational morphism $X/\delta \rightarrow Y$. Since Y is normal, this morphism is an isomorphism.

(3) 1-foliations and rational vector fields: A rational vector field $\delta \in \text{Der}_k k(X)$ is locally expressed as $\alpha \sum f_i \partial / \partial s_i$, where s_i are local coordinates, f_i are regular functions without common factors and $\alpha \in k(X)$. The divisor $\text{div}(\delta)$ associated to δ is defined by glueing the divisors $\text{div}(\alpha)$ on affine open sets. Then $\mathcal{O}_X(\text{div}(\delta))$ has a saturated p -closed invertible subsheaf structure of T_X :

$$\begin{array}{ccc} 0 \longrightarrow & \mathcal{O}_X(\text{div}(\delta)) & \xrightarrow{\cdot\delta} T_X \\ & h/\alpha & \longmapsto h \sum f_i \partial / \partial s_i \end{array}$$

where h is a regular function. Then we see that $\delta \mapsto \mathcal{O}_X(\text{div}(\delta))$ gives a one-to-one correspondence between p -closed rational vector fields modulo the equivalence and invertible 1-foliations.

Let X be a smooth projective surface over k and L an invertible 1-foliation of X . We have an exact sequence

$$0 \longrightarrow L \longrightarrow T_X \longrightarrow I_Z \otimes L' \longrightarrow 0,$$

where I_Z is the defining ideal sheaf of a zero-dimensional subscheme Z and L' is an invertible sheaf. We call the support of Z the *singular locus* of L and denote it by $\text{Sing } L$. Let Y be the Frobenius sandwich of degree p of X corresponding to L . Then $\text{Sing } Y = \pi(\text{Sing } L)$, where $\text{Sing } Y$ is the singular locus of Y , and outside $\text{Sing } L$ we have the canonical bundle formula $\omega_X \cong \pi^* \omega_Y \otimes L^{\otimes(p-1)}$; see [4].

Let $\delta \in \text{Der}_k k(X)$, and assume that δ is locally expressed as $\alpha(f\partial/\partial s + g\partial/\partial t)$ where s, t are local coordinates, f, g are regular functions without common factors and $\alpha \in k(X)$. Then $\text{Sing } \mathcal{O}_X(\text{div}(\delta))$ is defined locally by $f = g = 0$.

Globally F-regular varieties. Next we review generalities on globally F-regular varieties. See [12], [11] for further details. Let X be a projective variety over k . We say that X is *globally F-regular* if for any effective Cartier divisor D on X , there exists a power $q = p^e$ such that the composition map $\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \xrightarrow{F_*^e(s)} F_*^e \mathcal{O}_X(D)$ splits as an \mathcal{O}_X -module homomorphism, where s is a section of $\mathcal{O}_X(D)$ vanishing precisely along D .

Example 1.1. (1) Any projective toric variety is globally F-regular; see [12].

- (2) Let $\text{char } k = p > 5$. Then any smooth del Pezzo surface is globally F-regular; see [6].

Let X be a \mathbb{Q} -Gorenstein globally F-regular variety and H be an ample effective divisor. By the definition, there exists $e \geq 1$ such that the map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(H)$ splits as an \mathcal{O}_X -module homomorphism. On the other hand, we have $\mathcal{H}om_{\mathcal{O}_X}(F_*^e \mathcal{O}_X(H), \mathcal{O}_X) \cong F_*^e \mathcal{O}_X((1-p^e)K_X - H)$ by the adjunction formula. Thus a splitting $F_*^e \mathcal{O}_X(H) \rightarrow \mathcal{O}_X$ is its non-zero global section, so that there exists an effective divisor $D \sim (1-p^e)K_X - H$. This means that $-K_X$ is big. More generally, Schwede and Smith showed in [11] the following: If X is a globally F-regular variety, then there exists an effective \mathbb{Q} -divisor Δ on X such that the pair (X, Δ) is log Fano. This strong restriction on the structure of varieties is a motivation for the problem raised at the beginning.

The following lemma is a global variant of the well-known fact that a pure subring of a strongly F-regular ring is strongly F-regular. We include the proof for the reader's convenience; see [11] for a general case.

Lemma 1.2. *Let X be a globally F-regular variety over k and Y be an F^e -sandwich of X with the finite morphism $\pi : X \rightarrow Y$ through which the Frobenius morphism of X factors. Then Y is globally F-regular if and only if the associated ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ splits as an \mathcal{O}_Y -module homomorphism.*

Proof. Suppose that Y is globally F-regular. Then Y is F-split, i.e., the Frobenius ring homomorphism $\mathcal{O}_Y \rightarrow F_* \mathcal{O}_Y$ splits as an \mathcal{O}_Y -module homomorphism. Hence the Frobenius map $\mathcal{O}_Y \rightarrow F_*^e \mathcal{O}_Y$ splits as an \mathcal{O}_Y -module homomorphism. Now this map factors as $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_Y$, since Y is an F^e -sandwich of X . Thus the ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ splits as an \mathcal{O}_Y -module homomorphism.

Next suppose that the ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ splits. Let D be an effective Cartier divisor. Since X is globally F-regular, there exists $e \geq 1$ such that the map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(\pi^* D)$ splits as an \mathcal{O}_X -module homomorphism. Then we see that the composition map $\mathcal{O}_Y \rightarrow F_*^e \mathcal{O}_Y(D) \rightarrow \pi_* F_*^e \mathcal{O}_X(\pi^* D)$ splits as an \mathcal{O}_Y -module homomorphism from the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{\quad} & \pi_* \mathcal{O}_X \\ \downarrow & & \downarrow \\ F_*^e \mathcal{O}_Y(D) & \longrightarrow & \pi_* F_*^e \mathcal{O}_X(\pi^* D) \end{array}$$

Therefore $\mathcal{O}_Y \rightarrow F_*^e \mathcal{O}_Y(D)$ splits as an \mathcal{O}_Y -module homomorphism. \square

We will use the following lemma in the proof of the main theorem.

Lemma 1.3 ([7] Lemma 3.2). *Let $S = k[x, y]$ and let $\delta = f\partial/\partial x + g\partial/\partial y \in \text{Der}_k S$, where $f, g \in (x, y)$ and have no common factors. Suppose δ is p -closed. If the inclusion map $S^\delta \hookrightarrow S$ splits as an S^δ -module, then δ is not nilpotent.*

2. CLASSIFICATION OF GLOBALLY F -REGULAR F -SANDWICHES OF HIRZEBRUCH SURFACES

First we consider F -sandwiches of the projective plane \mathbb{P}^2 . Let X_0, X_1 and X_2 be homogeneous coordinates of \mathbb{P}^2 , i.e., $\mathbb{P}^2 = \text{Proj } k[X_0, X_1, X_2]$. Let $x = X_1/X_0, y = X_2/X_0$ (resp. $z = X_0/X_1, w = X_2/X_1$; $u = X_0/X_2, v = X_1/X_2$) be the affine coordinates of $U_0 := D_+(X_0)$ (resp. $U_1 := D_+(X_1)$; $U_2 := D_+(X_2)$).

Example 2.1. We give examples of F -sandwiches of \mathbb{P}^2 of degree p .

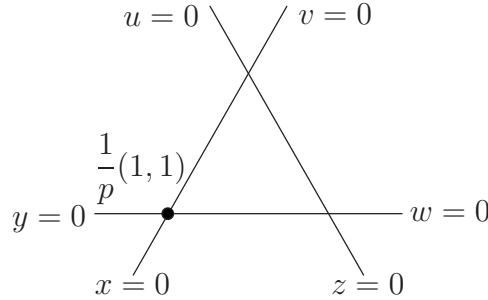
- (1) Let $\delta = x\partial/\partial x + y\partial/\partial y \in \text{Der}_k k(\mathbb{P}^2)$ and $\pi : \mathbb{P}^2 \rightarrow Y = \mathbb{P}^2/\delta$ the quotient map. If we express δ for the local coordinates z, w and u, v , we have

$$\delta = -z\partial/\partial z = -u\partial/\partial u.$$

Then the corresponding 1-foliation $\mathcal{O}_{\mathbb{P}^2}(\text{div}(\delta))$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$ and

$$\begin{aligned} \mathcal{O}_{\pi(U_0)} &= \mathcal{O}_{U_0}^\delta = k[x^p, x^{p-1}y, \dots, y^p], \\ \mathcal{O}_{\pi(U_1)} &= \mathcal{O}_{U_1}^\delta = k[z^p, w], \\ \mathcal{O}_{\pi(U_2)} &= \mathcal{O}_{U_2}^\delta = k[u^p, v]. \end{aligned}$$

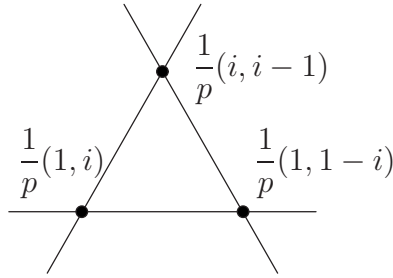
Hence Y has a toric singularity of type $\frac{1}{p}(1, 1)$ on $\pi(U_0)$. (We say a singularity (X, x) is a toric singularity of type $\frac{1}{p}(1, n)$, if the completion of $\mathcal{O}_{X, x}$ is isomorphic to $k[[x^i y^j \mid i + nj \equiv 0 \pmod{p}]]$.) We identify Y with \mathbb{P}^2 as a topological space via the homeomorphism $\pi : \mathbb{P}^2 \rightarrow Y$ on the underlying topological spaces. Then the configuration of the singular points of Y is as follows:



Let $\delta = x\partial/\partial x + iy\partial/\partial y \in \text{Der}_k k(\mathbb{P}^2)$ with $i \in \mathbb{F}_p, i \neq 0, 1$ and let $\pi : \mathbb{P}^2 \rightarrow Y = \mathbb{P}^2/\delta$ be the quotient map. For the local coordinates z, w and u, v , we have

$$\delta = -(z\partial/\partial z + (1-i)w\partial/\partial w) = -(iu\partial/\partial u + (i-1)v\partial/\partial v).$$

Hence the corresponding 1-foliation $\mathcal{O}_{\mathbb{P}^2}(\text{div}(\delta))$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}$, and $\pi(U_0)$, $\pi(U_1)$ and $\pi(U_2)$ have a toric singularity of type $\frac{1}{p}(1, i)$, $\frac{1}{p}(1, 1-i)$ and $\frac{1}{p}(i, i-1)$, respectively. The configuration of the singular points of Y is as follows:



Suppose $p = 2$. Let $\delta = x^2\partial/\partial x + y^2\partial/\partial y \in \text{Der}_k k(\mathbb{P}^2)$ and $\pi : \mathbb{P}^2 \rightarrow Y = \mathbb{P}^2/\delta$ the quotient map. For the local coordinates z, w and u, v , we have

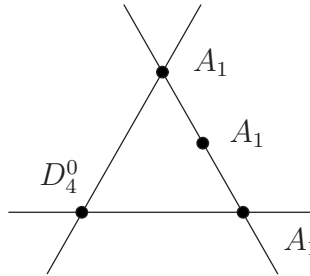
$$\delta = 1/z(z\partial/\partial z + w(w+1)\partial/\partial w) = 1/u(u\partial/\partial u + v(v+1)\partial/\partial v).$$

Hence the corresponding 1-foliation $\mathcal{O}_{\mathbb{P}^2}(\text{div}(\delta))$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)$, and $\pi(U_1)$ (resp. $\pi(U_2)$) has two A_1 -singularities at the points corresponding to $(0, 0), (0, 1) \in U_1$ (resp. $(0, 0), (0, 1) \in U_2$).

Since

$$\mathcal{O}_{\pi(U_0)} = \mathcal{O}_{U_0}^\delta = k[x^2, y^2, x^2y + xy^2] \cong k[X, Y, Z]/(Z^2 + X^2Y + XY^2),$$

we see that $\pi(U_0)$ has a D_4^0 -singularity (see [2] for rational double points in positive characteristic). The configuration of the singular points of Y is as follows:



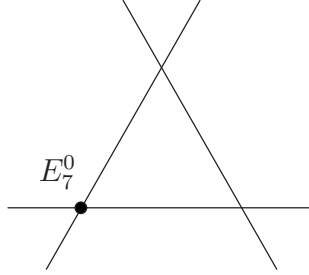
Suppose $p = 2$. Let $\delta = xy^2\partial/\partial x + (x^2 + y^3)\partial/\partial y \in \text{Der}_k k(\mathbb{P}^2)$ and $\pi : \mathbb{P}^2 \rightarrow Y = \mathbb{P}^2/\delta$ the quotient map. For the local coordinates z, w and u, v , we have

$$\delta = 1/z(w^2\partial/\partial z + \partial/\partial w) = 1/u((1 + uv^2)\partial/\partial u + v^3\partial/\partial v).$$

Hence the corresponding 1-foliation $\mathcal{O}_{\mathbb{P}^2}(\text{div}(\delta))$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)$, and Y is smooth on $\pi(U_1)$ and $\pi(U_2)$. Since

$$\mathcal{O}_{\pi(U_0)} = \mathcal{O}_{U_0}^\delta = k[x^2, y^2, x^3 + xy^3] \cong k[X, Y, Z]/(Z^2 + X^3 + XY^3),$$

we see that $\pi(U_0)$ has an E_7^0 -singularity. The configuration of the singular points of Y is as follows:



If X is a globally F -regular variety, then $\mathcal{O}_{X,x}$ is strongly F -regular for all $x \in X$ (see e.g., [12] for details). Now D_4^0 and E_7^0 -singularities are not strongly F -regular, so that F -sandwiches in the above example (3) and (4) are not globally F -regular. On the other hand, F -sandwiches Y in (1) and (2) are globally F -regular, since there exists a (global) splitting $1 - \delta^{p-1} : \pi_*\mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_Y$. Moreover we see that globally F -regular F -sandwiches of \mathbb{P}^2 of degree p are only those.

Let $N = \mathbb{Z}^2$ be a lattice with standard basis $e_1 = (1, 0), e_2 = (0, 1)$. For a fan Σ in $N \otimes \mathbb{R}$, we denote the associated toric variety over k by X_Σ . (For the general theory of toric varieties, we refer to [3].) For each $1 \leq i \leq p-1$, let Σ_i be the complete fan whose rays are spanned by $e_2, pe_1 - ie_2$ and $-pe_1 + (i-1)e_2$.

Theorem 2.2 ([7]). *A globally F -regular F -sandwich of \mathbb{P}^2 of degree p is isomorphic to either one of the singular toric surfaces X_{Σ_i} ($1 \leq i \leq p-1$). In particular, there are just $p-1$ isomorphism classes of globally F -regular F -sandwiches of \mathbb{P}^2 of degree p .*

In [7], we have seen that globally F -regular F -sandwiches of \mathbb{P}^2 have at most three singular points and given the classification by changing coordinates so that those singular points are located at the origins of three standard affine patches. A classification of globally F -regular F -sandwiches of Hirzebruch surfaces will be given by a similar argument,

that is, we will give the classification by changing coordinates so that their singular points are located at the origins of four standard affine patches.

The Hirzebruch surface \mathcal{H}_d of index $d \geq 0$ is the \mathbb{P}^1 -bundle associated to the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ on \mathbb{P}^1 . It is well-known that \mathcal{H}_d is the union of affine open sets U_i ($i = 1, 2, 3, 4$) whose affine coordinate rings are $k[x, y]$, $k[z, w] = k[x^d y, 1/x]$, $k[s, t] = k[1/x, 1/x^d y]$ and $k[u, v] = k[1/y, x]$, respectively. \mathcal{H}_d is a toric surface given by the complete fan whose rays are spanned by $\rho_1 = e_2$, $\rho_2 = e_1$, $\rho_3 = -e_2$ and $\rho_4 = -e_1 + de_2$. For each $i = 1, 2, 3, 4$, let D_i be the toric divisor that corresponds to the ray spanned by ρ_i .

Let Σ_{di} ($1 \leq i \leq p-1$) (resp. Σ_{d0} ; Σ_{dp}) be the complete fan whose rays are spanned by $e_2, pe_1 - ie_2, -e_2$ and $-pe_1 + (i+d)e_2$ (resp. $e_2, e_1, -e_2$ and $-pe_1 + de_2$; $e_2, e_1, -e_2$ and $-e_1 + dpe_2$).

First we consider the case where $d = 0$. Let $\delta \in \text{Der}_k(k(\mathcal{H}_0))$. Suppose that the corresponding 1-foliation $\mathcal{O}_{\mathcal{H}_0}(\text{div}(\delta))$ has a nonzero global section. Then we easily see that $\delta \sim (a_2 x^2 + a_1 x + a_0)\partial/\partial x + (b_2 y^2 + b_1 y + b_0)\partial/\partial y$, where $a_i, b_j \in k$. Let Y be a globally F-regular F -sandwich of \mathcal{H}_0 of degree p with the finite morphism $\pi : \mathcal{H}_0 \rightarrow Y$ through which the Frobenius morphism of \mathcal{H}_0 factors, and L be the corresponding 1-foliation. Then we have $\text{Hom}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_{\mathcal{H}_0}, \mathcal{O}_Y) \cong \pi_* L^{\otimes(p-1)}$ by the adjunction formula and the canonical bundle formula (see the proof of Theorem 2.4). Since a splitting of the associated ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{\mathcal{H}_0}$ is its nonzero global section, we see that L has a nonzero global section. Thus we may assume that the corresponding p -closed rational vector field δ is equivalent to $(a_2 x^2 + a_1 x + a_0)\partial/\partial x + (b_2 y^2 + b_1 y + b_0)\partial/\partial y$, where $a_i, b_j \in k$. By changing coordinates so that their singular points are located at the origins of U_i , we see that a globally F-regular F -sandwich of \mathcal{H}_0 is isomorphic to either one of the toric surfaces $X_{\Sigma_{0i}}$ ($0 \leq i \leq p-1$). In this case, there are just p isomorphism classes of the globally F-regular F -sandwiches.

In the case where $d \geq 1$, the situation is slightly complicated.

Lemma 2.3. *Suppose that $d \geq 1$. Let $\delta \in \text{Der}_k(k(\mathcal{H}_d))$. If the corresponding 1-foliation $\mathcal{O}_{\mathcal{H}_d}(\text{div}(\delta))$ has a nonzero global section, then*

$$\delta \sim (a_2 x^2 + a_1 x + a_0)\partial/\partial x + (F(x)y - da_2 x + b)y\partial/\partial y,$$

where $a_i, b \in k$ and $F(x) \in k[x]$ with $\deg F(x) \leq d$.

Proof. Let $\delta = f\partial/\partial x + g\partial/\partial y$. If $f = 0$ or $g = 0$, then $\delta \sim \partial/\partial y$ or $\partial/\partial x$, so there is nothing to prove. Suppose that $f \neq 0$ and $g \neq 0$.

Multiplying by a suitable rational function on \mathcal{H}_d , we have

$$\delta \sim \sum_{0 \leq i,j} a_{ij} x^i y^j \frac{\partial}{\partial x} + \sum_{0 \leq m,n} b_{mn} x^m y^n \frac{\partial}{\partial y},$$

where $a_{ij}, b_{mn} \in k$, and $\sum_{0 \leq i,j} a_{ij} x^i y^j$ and $\sum_{0 \leq m,n} b_{mn} x^m y^n$ have no common factors. If we express δ for the local coordinates u, v , then we see that the coefficient of δ in $\partial/\partial u$ (resp. $\partial/\partial v$) equals $-\sum_{0 \leq m,n} b_{mn} v^m / u^{n-2}$ (resp. $\sum_{0 \leq i,j} a_{ij} v^i / u^j$). If $b_{mn} \neq 0$ for some $n \geq 3$ or $a_{ij} \neq 0$ for some $j \geq 1$, then $\text{ord}_{D_3} \text{div}(\delta) < 0$. Since $\sum_{0 \leq i,j} a_{ij} x^i y^j$ and $\sum_{0 \leq m,n} b_{mn} x^m y^n$ have no common factors, this means that $H^0(\mathcal{H}_d, \mathcal{O}_{\mathcal{H}_d}(\text{div}(\delta))) = 0$, which is a contradiction. Therefore we have $b_{mn} = 0$ for $n \geq 3$ and $a_{ij} = 0$ for $j \geq 1$. Thus we have $\delta \sim \sum_{0 \leq i} a_{i0} x^i \partial/\partial x + \sum_{0 \leq m, 0 \leq n \leq 2} b_{mn} x^m y^n \partial/\partial y$. We denote the right-hand side by δ' .

If we express δ' for the local coordinates s, t , then we see that the coefficient of δ' in $\partial/\partial s$ equals $-\sum_{0 \leq i} a_{i0} / s^{i-2}$. If $a_{i0} \neq 0$ for some $i \geq 3$, then $\text{ord}_{D_3} \text{div}(\delta) \leq 0$ and $\text{ord}_{D_4} \text{div}(\delta) < 0$. This means that $H^0(\mathcal{H}_d, \mathcal{O}_{\mathcal{H}_d}(\text{div}(\delta))) = 0$, which is a contradiction. Therefore we have $a_{i0} = 0$ for $i \geq 3$. Thus we have $\delta' = (a_{20}x^2 + a_{10}x + a_{00})\partial/\partial x + (F_2(x)y^2 + F_1(x)y + F_0(x))\partial/\partial y$, where $F_i \in k[x]$. Considering δ' for the local coordinates s, t again, we see that $\deg F_2(x) \leq d$, $F_1(x) = -da_{20}x$ and $F_0(x) = 0$. This means that $\delta \sim (a_2x^2 + a_1x + a_0)\partial/\partial x + (F(x)y - da_2x + b)y\partial/\partial y$, where $a_i, b \in k$ and $F(x) \in k[x]$ with $\deg F(x) \leq d$. \square

Theorem 2.4. *A globally F -regular F -sandwich of the Hirzebruch surface \mathcal{H}_d of index $d \geq 1$ of degree p is isomorphic to either one of the toric surfaces $X_{\Sigma_{di}}$ ($0 \leq i \leq p$). In particular, there are just $p+1$ isomorphism classes of globally F -regular F -sandwiches of \mathcal{H}_d of degree p .*

Proof. Let Y be a globally F -regular F -sandwich of \mathcal{H}_d of degree p with the finite morphism $\pi : \mathcal{H}_d \rightarrow Y$ through which the Frobenius morphism of \mathcal{H}_d factors, and let $L \subset T_{\mathcal{H}_d}$ (resp. $\delta \in \text{Der}_k k(\mathcal{H}_d)$) be the corresponding 1-foliation (resp. the p -closed rational vector field). Since the associated ring homomorphism $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{\mathcal{H}_d}$ splits by Lemma 1.2, there is a nonzero \mathcal{O}_Y -module homomorphism $\pi_* \mathcal{O}_{\mathcal{H}_d} \rightarrow \mathcal{O}_Y$. Outside $\text{Sing } L$ we have

$$\begin{aligned} \text{Hom}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_{\mathcal{H}_d}, \mathcal{O}_Y) &\cong \text{Hom}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_{\mathcal{H}_d}, \omega_Y) \otimes \omega_Y^{-1} \\ &\cong \pi_*(\omega_{\mathcal{H}_d}) \otimes \omega_Y^{-1} \\ &\cong \pi_*(\omega_{\mathcal{H}_d} \otimes \pi^*(\omega_Y^{-1})) \\ &\cong \pi_*(L^{\otimes(p-1)}), \end{aligned}$$

which gives a (global) isomorphism $\mathrm{Hom}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_{\mathcal{H}_d}, \mathcal{O}_Y) \cong H^0(\mathcal{H}_d, L^{\otimes(p-1)})$ since Y is normal. Thus $L^{\otimes(p-1)}$ has a nonzero global section. Since L is a line bundle on \mathcal{H}_d , L has a nonzero global section.

If Y is singular at a point on the image of the negative section D_1 , after a suitable change of coordinates, we may assume that Y is singular at the point corresponding to the origin of U_1 . (In what follows we refer to this assumption as “the assumption for the arrangement of singularities”.)

Then by Lemma 2.3 we may assume that

$$\delta = (a_2x^2 + a_1x + a_0)\partial/\partial x + (F(x)y - da_2x + b)y\partial/\partial y,$$

where $a_i, b \in k$ and $F(x) \in k[x]$ with $\deg F(x) \leq d$. We will proceed by changing coordinates so that singular points are located at the origins of U_i .

We divide the remainder of the proof into three steps:

(1) Suppose that $a_2 = a_1 = 0$, i.e., $\delta = a_0\partial/\partial x + (F(x)y + b)y\partial/\partial y$. If $a_0 = 0$ then $\delta \sim \partial/\partial y$. Suppose that $a_0 \neq 0$. Then we may assume that $\delta = \partial/\partial x - (F(x)y + b)y\partial/\partial y$. If we express δ for the local coordinates u, v , we have $\delta = \partial/\partial v - (F(v) + bu)\partial/\partial u$. Since δ is p -closed, we see that $b = 0$ and the $(p-1)$ -th derivative $F^{(p-1)}(v) = 0$. In particular, $\delta = \partial/\partial x - F(x)y^2\partial/\partial y$. For the local coordinates z, w , we have $\delta = z(dw - zF(1/w)w^d)\partial/\partial z - w^2\partial/\partial w$. If $\deg F = d$, then the constant term of $F(1/w)w^d$ is not zero, so that Y is singular at the point corresponding to the origin of U_2 . On the other hand, Y is smooth at the point corresponding to the origin of U_1 , since $\delta = \partial/\partial x - (F(x)y + b)y\partial/\partial y$ for the local coordinates x, y . This contradicts the assumption for the arrangement of singularities. Thus $\deg F \leq d-1$. Then there exists a polynomial $G(v) \in k[v]$ such that $\deg G(v) \leq d$ and its derivative equals $F(v)$ since the $(p-1)$ -th derivative $F^{(p-1)}(v) = 0$. After a change of a coordinate $u + G(v) \mapsto u$, we have $\delta = \partial/\partial v$. For the local coordinates x, y , we have $\delta = \partial/\partial x$.

(2) Suppose that $a_2 = 0$ and $a_1 \neq 0$. Then we may assume that $\delta = (x + a_0)\partial/\partial x + (F(x)y + b)y\partial/\partial y$. If $(x - a_0)|(F(x)y + b)$, then $(x - a_0)|F(x)$ and $b = 0$. This means that $\delta \sim \partial/\partial x - G(x)y^2\partial/\partial y$, where $G(x) = F(x)/(x - c)$, and this is the case (1). Now we assume that $(x - a_0) \nmid (F(x)y + b)$. From the assumption for the arrangement of singularities we see that $a_0 = 0$, i.e., $\delta = x\partial/\partial x + (F(x)y + b)y\partial/\partial y$. For the local coordinates u, v , we have $\delta = v\partial/\partial v + (F(v) + du)\partial/\partial u$. Since δ is p -closed, we see that

$$\delta = v\partial/\partial v + \left(\sum_{\substack{0 \leq j \leq d, \\ j \not\equiv i \pmod{p}}} c_j v^j + iu \right) \partial/\partial u,$$

where $i \in (\mathbb{Z}/p\mathbb{Z})^\times$, $c_j \in k$. After a change of a coordinate

$$u - \sum_{\substack{0 \leq j \leq d, \\ j \not\equiv i \pmod{p}}} c_j v^j / (j - i) \mapsto u,$$

we have $\delta = v\partial/\partial v + iu\partial/\partial u$. For the local coordinates x, y , we have $\delta = x\partial/\partial x - iy\partial/\partial y$.

(3) Suppose that $a_2 \neq 0$. Then we may assume that $\delta = (x^2 + a_1x + a_0)\partial/\partial x + (F(x)y - dx + b)y\partial/\partial y$. If $(x - A)|(x^2 + a_1x + a_0)$ and $(x - A)|(F(x)y - dx + b)$ for some $A \in k$, then we have $\delta \sim (x - B)\partial/\partial x + (G(x)y - b)y\partial/\partial y$, where $B \in k$ is the other root of $x^2 + a_1x + a_0 = 0$ and $G(x) = F(x)/(x - A) \in k[x]$. This is the case (2). We assume that $x^2 + a_1x + a_0$ and $F(x)y - dx + b$ have no common factor. Then from the assumption for the arrangement of singularities we see that $a_0 = 0$. If $a_1 = 0$, then $\delta = x^2\partial/\partial x - (F(x)y - dx + b)y\partial/\partial y$. If $x \nmid (F(x)y - dx + b)$, then δ must be nilpotent, which is a contradiction. Therefore $x|(F(x)y - dx + b)$. Then $\delta \sim x\partial/\partial x - (G(x)y - b)y\partial/\partial y$, where $G(x) = F(x)/x \in k[x]$. This is the case (2). Now we assume that $a_1 \neq 0$. If we express δ for the local coordinates z, w , we have $\delta = (F(1/w)w^d z^2 + (da_1 + b)z)\partial/\partial z - (1 + a_1w)\partial/\partial w$, where $G(w) = F(1/w)w^d \in k[w]$. After a change of a coordinate $w + a_1^{-1} \mapsto w$, we have $\delta = (G(w - a_1^{-1})z^2 + (da_1 + b)z)\partial/\partial z - a_1^{-1}w\partial/\partial w$. For the local coordinates x, y , we have $\delta \sim a_1^{-1}x\partial/\partial x + (H(x)y + b)y\partial/\partial y$, where $H(x) \in k[x]$. This is the case (2).

Therefore we see that $\delta \sim x\partial/\partial x + iy\partial/\partial y$ ($i \in \mathbb{Z}/p\mathbb{Z}$) or $\partial/\partial y$. Then we can easily check that \mathcal{H}_d/δ is isomorphic to either one of the toric surfaces $X_{\Sigma_{di}}$ ($0 \leq i \leq p$). \square

Remark 2.1. Ganong and Russell showed in [5] that for each Hirzebruch surface there are at most two smooth F -sandwiches. The smooth F -sandwich is given by the quotient of \mathcal{H}_d by the rational vector field $\partial/\partial x$ or $\partial/\partial y$.

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DEPARTMENT OF GENERAL EDUCATION, FUKUSHIMA NATIONAL COLLEGE OF TECHNOLOGY, 30 AZA-NAGAO, KAMIARAKAWA, IWAKI-SHI, FUKUSHIMA 970-8034, JAPAN

E-mail address: sawada@fukushima-nct.ac.jp